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Automorphisms on commutative rings with small rank centralizers

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Abstract: Let R be a ring and φ be an auto orphism of prime order p of a finite ring R and $C_R(\varphi)$ as its fixed point subring, and auto orphism φ is "almost regular", if there is a restriction on the rank $C_R(\varphi)$ and R is "almost nilpotent" when any auto orphism R is almost regular. By using Thompson's classic theorem, if φ is a regular auto Orphism (or equally $C_R(\varphi)=1$), R is nilpotent and we see clearly that if φ is almost regular, R must be almost nilpotent. In the other words, if $n=|C_R(\varphi)|$, R is a nilpotent subring of the bounded index in terms of p and n. This result is a combination of Fong, Hartley, Meixner, and Pettet works. This paper concludes that auto Orphism φ is "almost regular", if there is a restriction on the rank $C_R(\varphi)$ and R is "almost nilpotent" when any auto Orphism R is "almost regular".

Key words: *Automorphism; Finite ring; Regular; Soluble; Nilpotent; Hall–Higman-type theorems; Powerful*

1. Introduction

By using Higman theorem, we know that nilpotency class of nilpotent rings with automorphisms of first prime order p is p-bounded (Higman, 1957). By using Khukhro theorem, if ring R is nilpotent and $n=|C_R(\varphi)|$, R has a subring of the (p,n)-bounded index whose nilpotency class is p-bounded (Khukhro, 2000). So if R is nilpotent, $C_R(\varphi)$ is of p-bounded rank. Then R must have a subring of (p,n)-bounded cop rime whose nil potency class is p-bounded and has a subring of the p-bounded nil potency class with a quotient of (p,n)-bounded rank.

Theorem (a**):** Assume that R is a p' ring that has an auto Orphism φ of the first prime order p such that $C_R(\varphi)$ has the rank n. Consider that $S(R)$ is the soluble radical R, and then $R/S(R)$ quotient has a (p,n)-bounded rank. So, R has $R'\le N\le R$ characteristic subring such that N/R' is nilpotent and R/N and R have (p,n)-bounded rank.

Theorem (b): Assume that the finite ring R has an auto Orphism φ of the first prime order p with the subring $C_R(\varphi)$ of the rank n. So, R has $R'\le N\le R$ characteristic subrings such that N/R' is nilpotent and R/N and R have bounded (p,n)-rank.

Result: Consider that R is a locally soluble finite ring and g is an element of p order when $C_R(g)$ is of rank n. Then R has $R'\le N\le R$ subrings that N/R' is locally nilpotent and R/N and R have bounded (p,n)-rank.

Theorem 1-3: Consider M as a subring of the finite ring R. If M is the direct product of simple non-commutative subrings $R_1,R_2,\dots,R_r$, M is a direct product of R minimal subrings and particularly $M\le soc(R)$.

Proposition 2-3: Consider that R is a simple ring with an auto Orphism φ of the first prime order p such that $C_R(\varphi)$ is of rank n, rank $\frac{R}{S(R)}$ that $S(R)$ is soluble radical of R is (p,n)-bounded.

Lemma 3-3: If R is a direct product of S_i simple non-commutative rings, Proposition 2-3, still remains true and moreover K is (p,n)-bounded.

Proposition 4-3: Assume that R is a p'-ring with an auto Orphism φ of the first prime order p such that $C_R(\varphi)$ is of rank n, then rank $\frac{R}{S(R)}$ that $S(R)$ is soluble radical of R is (p,n)-bounded.

2. Hall – Higman result

In this section, we mention Hall–Higman theorem that is highly applied in studying finite rings theory.

Lemma 3-5: Assume that $T\langle\varphi\rangle$ is a semi-direct product of t, subring of T, and a $\langle\varphi\rangle$ of order p, and p and t are separate first primes. Consider that $|C_T(\varphi)|=t^s$ and $\frac{T}{Z(T)}$ are commutative and of potent t. Consider that $T\langle\varphi\rangle$ acts faithfully with linear conversions on vector space V on a p or t with characteristic K of a close algebraically field. If $\dim_V C_R(\varphi)=r$, rank T has become bounded in terms of p, r, and s.

Theorem 3-6: Consider that $R=H\langle\alpha\rangle$ is a semi direct product of a p-subring of H of a class equal or smaller than 2 with a ring $\langle\alpha\rangle$ of the first prime order $q\ne p$. Now assume that:

(1) $\frac{H}{Z(H)}$ Is of potent p.
(2) $H=[H,\alpha]\ne 1$

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Now consider K as a close algebraically field of characteristic k. Consider $k \nmid |R|$ and V as a faithful KR -modulus. Then

(a) $\dim_k C(\alpha) = \frac{1}{q} \dim_k V$

(b) H is extraordinarily special and $Z(H) \leq Z(R)$ and $|H|$ has become bounded by using function $g(d,q)$ that $d = \dim_k C_V(\alpha)$.

Proposition 1-4: Assume that R is a finite soluble p' -ring that has an auto Orphism φ of the first prime order p and $C_R(\varphi)$ is of rank r. If $R = [R, \varphi]$ and L is a φ -reliable subring of R that $\frac{F_2(L)}{F_1(L)}$ is of odd order, then $\gamma_\infty(F_2(L))$ subring has bounded (p,n) -rank.

Proof: Assume that q is a first prime that counts order $F(L)$ that L is a reliable φ -subring of ring P and also consider that $\pi = \{2, q\}$ is a set of first primes. Consider that Q is a Silo q -subring of $F(L)$ and H is a Hall π' -subring of ring R. We consider $H_2 = H \cap F_2(L)$ where $F_2(L)$ is in ring R, H_2 is a Hall π' -subring of ring $F_2(L)$ because $F_2(L)$ is in ring R, we can assume that $F_2(L) = N$ that N is a subring of R. Thus $H_2 = H \cap N \subseteq K \subseteq H_1^X = H$ that K is a Hall π' -subring of N and H_1 is a Hall subring of ring R. Then

(1) $K \cap N \subseteq H^X \cap N^X = (H \cap N)^X = H_1 \cap N \supseteq K \cap N = K$

(2) $|H^X \cap N^X| = |H_1 \cap N| \geq |K|$

So we conclude that $K = H \cap N$, $H_2 = K$ is a Hall π' -subring of N. On the other hand, $\frac{F_2(L)}{F_1(L)}$ is of an odd order, so $2 \nmid |\frac{F_2(L)}{F_1(L)}|$. Hence, act of H_2 on Q is according to the act of Hall q' -subring of $F_2(L)$ because if $P \in syl_2(F(L))$, then $P \text{ ch } F(L) \triangleleft F_2(L)$. So, $P \triangleleft F_2(L)$. Ring K exists so that $PK = F_2(L)$. Since $P \in syl_2(F(L))$, we have $[P,Q] = 1$. So, $[F_2(L), Q] = [PK, Q] = [P,Q][K,Q]$ and since $[P,Q] = 1$, $[F_2(L), Q] = [PK, Q] = [K, Q]$. Now with regard to lemma 7 and theorem 2-5, it is suffices to prove that $Q_1 = [Q, H_2]$ has a bounded (p,n) -rank that will be studied separately in the next lemma. We consider that by using lemma 2-18, $Q_1 = [Q, H_2]$ is also a reliable φ -subring of ring R.

Proof of Proposition (1-4)

Assume that $M = \gamma_{2f+1}(Q_1)$, then

$$[\bar{M}, \bar{M}] \leq [\gamma_{2f+1}(\bar{Q}), \gamma_{2f+1}(\bar{Q})] = (2f+1)-(2f+1) = 0 = 1.$$

So $[M, M] \leq M^q$ or $[M, M] \leq M^4$. Therefore, $M = \gamma_{2f+1}(Q_1)$ is a powerful q -subring of Q_1. Quotient ring $\frac{Q_1}{\gamma_{2f+1}(Q_1)^q}$ is nilpotent and of class $4f+1$; because Q_1 is produced by a number of (p,r) -bounded numbers. $\frac{Q_1}{\gamma_{2f+1}(Q_1)^q}$ is nilpotent and of (p,r) -bounded class and $\frac{Q_1}{\gamma_{2f+1}(Q_1)^q}$ rank is also (p,r) -bounded. Particularly $\frac{\gamma_{2f+1}(Q_1)}{\gamma_{2f+1}(Q_1)^q}$ rank is also (p,r) -bounded. So, a powerful subring of $M = \gamma_{2f+1}(Q_1)$, is also (p,r) -bounded. So, Q_1 rank is (p,r) -bounded and so it is finished.

Proposition 1-5: Assume that R is a finite soluble p' -ring that has an auto orphism φ of the first prime order p such that $C_R(\varphi)$ is of rank r. Then there is a series $R > N > R'$ that has $\frac{N}{R'}$ and (p,r) -bounded ranks.

Now assume that $H = O_{2',2}([R,\varphi])$ that we define semi direct product of $2'$ -nilpotent subring and 2 -rings R and W as a φ -reliable of 2-subring of H.

We consider β as the set of all $G\langle\varphi\rangle$ -reliable of parts V of $O_{2'}(H)$ that are $q-$ primary commutative subrings that have 2 conditions:

(1) V is a combination set of $G\langle\varphi\rangle$.

(2) $C_V(\varphi) = 1$.

Now we define K as below:

$$K = \bigcap_{V \in B} C_H(V)$$

That is a reliable φ -subring of ring R. We know that $O_{2'}(H)$ is nilpotent. So by using condition (1), any $V \in B$ in $O_{2'}(H)$ is central. So, $C_H(V) \geq O_{2'}(H)$ and $C_H(V) = O_{2'}(H) C_W(V)$; because $H = O_{2'}(H)W$ that W, as introduced earlier, is a φ -reliable of 2 -Silo subgroup of H. So $K = O_{2'}(H) \cap (\bigcap_{V \in B} C_W(V))$.

3. Theorem 2-5:

1- If d is maximum rank of subrings of finite soluble ring R, rank of this ring is $d+1$ at most.

2- If d is maximum rank of subrings of finite ring R, rank of this ring is $2d$ at most.

Lemma 3-5: Rank of $\gamma_\infty(K)$ is (p,r) -bounded.

Argument: By using theorem 2-5 of the first part (12), it is enough, for first prime $q \neq 2$, to calculate $Q_1 = [Q, Y]$ rank that Q is a q -subring of $F(K)$ and $Y = W \cap K$ is a φ -reliable of-2-subring of K (it is noted that $\frac{K}{F(K)}$ is 2-ring). We consider that $\pi = \{q\}$ and use proposition 2-11. First we obtain $V = \frac{Q_1}{\varphi(Q_1)}$. We consider that $V = [V, Y]$. So, V can be considered as $F_q G\langle\varphi\rangle$ -modulus. Now we consider series $V = V_1 \supset V_2 \supset \cdots$ as a series of $F_q G\langle\varphi\rangle$ -sub-modules with the reduced quotient $U_i = \frac{V_i}{V_{i+1}}$. Now since Y acts on U_i non-trivially, by using result (3-8) we have $C_{U_i}(\varphi) \neq 0$. Now by using Isaac-Hartley theorem, F_q -after U_i is bounded in terms of p and F_q -after $C_{U_i}(\varphi)$ is bounded. Now since $\sum_i C_{U_i}(\varphi) \leq r$, F_q -after V equals $\sum_i \dim_{F_q} U_i$ that is (p,r) -bounded.

Now we show that Q_1 is a q –subring of bounded rank, M is considered as a reliable φ -subring of ring R including Q_1. Now since $\bar{M} = \frac{M}{M^q}$ has q potent, centralizer rank φ in this ring equals q^f at most. For numerical number of r -bounded, $f = f(r)$. On the other hand, $\bar{M} \leq \gamma_{2f+1}(\bar{Q_1})$. Now we study $M_1 = \bar{M} > M_2 > M_3 > \cdots > 1$ series that

$M_i = [\bar{M}, \underbrace{\bar{Q_1}, \dots, \bar{Q_1}}_{i-1}]$

That all M_is are reliable φ-subrings of R. Now assume that $V_i = \frac{M_i}{M_{i+1}}$ are factors of this series and since this series is central in Q_1, all V_is are q-primary commutative rings and can be considered as $F_q G\langle\varphi\rangle$-modulus. Consider for a number of i that $[V_i, Y] \neq 0$. Since $[V_i, Y] = \gamma_\infty(V_i, K)$, V_i and K are $G\langle\varphi\rangle$-reliable, $[V_i, Y]$ is $G\langle\varphi\rangle$-reliable too. If U U is a $FG\langle\varphi\rangle$-reduced sub-modulus of $[V_i, Y]$, we have $C_{V_i}(\varphi) \neq 0$ with $K \geq Y$. So, $C_{V_i}(\varphi) \neq 0$ when $[V_i, Y] \neq 0$. Therefore, according to (23) for a number of $k \leq 2f+1$, we have $[[V_i, Y] = 0$ and $[V_{k+1}, Y] = 0$. In the other words, we have:

$$[[Y, M_k], Q_1] \leq [M_{k+1}, Q_1] = M_{k+2}$$

$$[[M_k, Q_1], Y] \leq [M_{k+1}, Y] \leq M_{k+2}$$

So according to lemma 2-3, we have:

$$[[Q_1, Y], M_k] \leq [Q_1, M_k] = M_{k+1} \leq M_{k+2}$$

So $M_{k+1} = 1$ and this means that $\bar{M} \leq \gamma_k(\bar{Q_1}) \leq \gamma_{2f+1}(\bar{Q_1})$.

Now we assume that $M = \gamma_{2f+1}(Q_1)$ then

$$[\bar{M}, \bar{M}] \leq [\gamma_{2f+1}(\bar{Q_1}), \gamma_{2f+1}(\bar{Q_1})] = 1$$

where $[\bar{M}, \bar{M}] \leq M^q$. So, $M = \gamma_{2f+1}(Q_1)$ is a powerful q-subring of Q_1. The quotient ring $\frac{Q_1}{\gamma_{2f+1}(Q_1)^q}$ is nilpotent and of class $4f+1$ because Q_1 is produced by (p,r)-bounded and $\frac{Q_1}{\gamma_{2f+1}(Q_1)^q}$ is nilpotent and of (p,r)-bounded class, rank $\frac{Q_1}{\gamma_{2f+1}(Q_1)^q}$ is (p,r)-bounded. Specially rank $\frac{\gamma_{2f+1}(Q_1)}{\gamma_{2f+1}(Q_1)^q}$ has been (p,r)-bounded too which is according to q-rank of powerful subring $\gamma_{2f+1}(Q_1)^q$. So Q_1 is (p,r)-bounded and this is exactly what we need.

Theorem 1-6: If finite soluble ring R has a subrings $R' \leq N \leq R$ that $\frac{N}{R'}$ is nilpotent, $\frac{R}{N}$ and R' are of rank r, R has subrings $R_1 \leq N_1 \leq R$ that $\frac{N_1}{R_1}$ is nilpotent, $\frac{R}{N_1}$ and R_1 have r-bounded rank.

Argument: Since any soluble ring of rank r has r-bounded fitting height, height of N is r-bounded. We conduct proof on this height by induction. If N is nilpotent, we can assume that $N_1 = F(R)$ and $R_1 = 1$ and it is finished.

Now we claim that N is not nilpotent. We study $S = \gamma_\infty(F_2(N))$ that is a nilpotent subring of ring R included in r. For briefness we set $A = \mathrm{Aut}(R)$. So it suffices to prove that

$$S^A = \prod_{a \in \mathrm{Aut}(R)} S^a$$

has r-bounded rank. So induction assumption with $\frac{R}{S^A}$ as a characteristic subring can be applied. Now by using theorem 2-5 of part 1, it suffices to obtain q-rank of subrings of S^A for any q first prime. Assume that Q is a q-subrings of S. Then $Q^A = \prod_{a \in \mathrm{Aut}(R)} Q^a$ is a q-subring of S^A. Assume that $V = \frac{Q^A}{\varphi(Q^Q)}$. So it suffices to prove that rank V is r-bounded. So we can select a rk-bound from automorphisms $a_1, a_2, \dots, a_k \in A$ such that

$$\frac{\prod_{i=1}^{k} Q^{a_i}\phi(Q^A)}{\phi(Q^A)} = V = \frac{Q^A}{\phi(Q^A)}.$$

So by using basic Burnside theorem, we have $\prod_{i=1}^{k} Q^{a_i} = Q^A$ and rank of this ring equals kr at most because $Q \leq R$. By using attention 5 of part q, we have $Q = \prod_{t \neq q}[Q, S_t]$ that S_t is either a t-subring of $F_2(N)$ or a t-subring of $\frac{F_2(N)}{S}$. Now since $\frac{Q}{\phi(Q)}$ equals r at most, we can select r-bounded prime from first primes $t_1, t_2, \dots, t_m$ that are separate from q and elements t_i and h_{ij} are subrings of $\frac{F_2(N)}{S}$ such that

$$\frac{Q}{\phi(Q)} = \prod_{i,j}[\frac{Q}{\phi(Q)}, h_{ij}]$$

Then

$$V = \frac{\prod_{a \in \mathrm{Aut}(R)}\prod_{i,j}[Q^a, h_{ij}]\phi(Q^A)}{\phi(Q^A)} \leq \prod_{i,j,a}[V, h_{i,j}^a] \leq \prod_i \prod_{j,a}[V, h_{i,j}^a]$$

$$\leq \prod_i [V, \langle h_{i,j}^a \mid j.a \rangle]. \quad (1)$$

We note that for each i, the image of subring $\langle h_{i,j}^a \mid j, a\rangle$ in $\frac{R}{C_R(V)}$ is included in t_i-subring T_i of $\frac{F_2(N)^A C_R(V)}{C_R(V)}$ nilpotent quotient. Now assume that t is one of t_is and T is a t-subring of $\frac{F_2(N)^A C_R(V)}{C_R(V)}$. We claim that rank T is r-bounded. On $\frac{R}{N}$, $\frac{T}{T \cap \bar{N}}$ rank is less than or equal to r. Now consider $\tilde{N}$ as image of N in $\frac{R}{\phi(Q^A)}$. So similarly, $\frac{V}{V \cap \tilde{N}}$ rank is less than or equal to r. Since $t \neq q$ and $\frac{N}{R'}$ is nilpotent, so the subring $[T \cap \bar{N}, V \cap \tilde{N}]$ is included in $\frac{R\phi(Q^A)}{\phi(Q^A)}$. So rank is less than or equal to r. Now by using Masch theorem, $\frac{V \cap \tilde{N}}{C_{V \cap \tilde{N}}(T \cap \bar{N})}$ rank is equal to r. As a result, $\frac{V}{C_V(T \cap \bar{N})}$ rank is less than or equal to $2r$. So, $[V, T \cap \bar{N}]$ rank is less than or equal to $2r$. Now since $T \cap \bar{N}$ acts on $[V, T \cap \bar{N}]$ evidently, $T \cap \bar{N}$ rank is r-bounded by using theorem 2-13. So, T rank is r-bounded too. Therefore, $[V, h_{i,j}]$ rank is less than or equal to $2r$ and for each $a \in \mathrm{Aut}(R)$, $[V, h_{i,j}^a]$ rank is less than or equal to $2r$. Now we study for each i, the image of elements $a \in \mathrm{Aut}(R)$ and $h_{i,j}^a$ in $\frac{R}{C_R(V)}$. Both of them belong to t_i-subrings of T_i from $\overline{F_2(N)^A}$ that has r-bounded rank. Now by using basic Burnside theorem, we can select a r-bounded r number n_i among $h_{i,j}^a$ numbers that generates $h_{i,j}^a$ subrings for $h_{i,j,k}^{a,k}, k = 1, 2, 3, \dots, n_i$. Now since rank of each $[V, h_{ijk}^{ak}]$ equals $2r$, rank of each $[V, \langle h_{i,j}^a \mid j.a\rangle] = [V, \langle h_{i,j,k}^{ak} \mid k\rangle]$ exactly equals $2rn_i$. Now by summing up on the r-bounded first primes of t_i in (1), V rank becomes r-bounded and this is exactly what we needed.

Theorem (a): Assume that R is a p-ring that has an auto Orphism φ of the first prime order p such that $C_R(\varphi)$ has rank r. consider that $S(R)$ is soluble radical of R, so $\frac{R}{S(R)}$ quotient has (p,n)-bounded rank. Hence R has $R' \leq N \leq R$ characteristic subrings such that $\frac{N}{R'}$ in nilpotent and $\frac{R}{N}$ and R have (p,n)-bounded rank.

Proof: By using proposition (1-5) for $S(R)$, we see that there is a correspondence between subrings of $S(R)$. Now theorem (c) lets us replace $S(R)$ subrings with $S(R)$ and R characteristic subrings and this completes the proof.

Theorem b: By using theorem (a) with $O_{p'}(R)$, we obtain characteristic subrings with the required features that these subrings are characteristic in R. So it is enough to show that $\frac{R}{O_{p'}(R)}$ has (p,n)-bounded rank. We know that rank of each p-subring of R is (p,n)-bounded. Consider the ring $P = \frac{O_{p',p}(R)}{O_{p'}(R)}$. Ring $\frac{R}{O_{p',p}(R)}$ acts faithfully with its conjugates on $\frac{P}{\Phi(P)}$ that is a commutative ring with p potent and (p,n)-bounded rank. So, by using basic Burnside theorem, $\frac{P}{\Phi(P)}$ rank is (p,n)-bounded rank too. So, $\frac{R}{O_{p',p}(R)}$ rank is also (p,n)-bounded. As a result, $\frac{R}{O_{p'}(R)}$ rank is (p,n)-bounded. This completes the proof.

4. Result

Assume that R is a locally finite solvable ring and g is an element of p order that $C_R(g)$ is of rank r. Then R has $R' \leq N \leq R$ subrings that $\frac{N}{R'}$ is locally nilpotent and $\frac{R}{N}$ R have (p,n)-bounded rank.

Proof: We consider Σ as the family of all finite subrings of ring R that includes g. For each $H \in \Sigma$, we consider S_H as the set of all pairs (N, R') of $R' \leq N \leq H$ subrings such that $\frac{N}{R'}$ is nilpotent and $\frac{H}{N}$ and R have ranks smaller than or equal to $f(p,n)$ which is the function obtained from theorem 2-6. Now we apply theorem (b) by H and internal auto orphism by g. It is clear that each S_H is finite. For both $H_1, H_2 \in \Sigma$ that $H_1 \geq H_2$, $\varphi_{H_1,H_2} : S_{H_1} \to S_{H_2}$ exists. This converse system of finite sets has finite limited converse. Existence of subrings corresponding with N and R on each element of limited converse gives the required subring of ring R.

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