Automorphisms on commutative rings with small rank centralizers

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Abstract: Let $\mathcal{R}$ be a ring and $\phi$ be an auto-orphism of prime order $p$ of a finite ring $\mathcal{R}$ and $C_x(\phi)$ as its fixed point subring, and auto-orphism $\phi$ is "almost regular", if there is a restriction on the rank $C_x(\phi)$ and $\mathcal{R}$ is "almost nilpotent" when any auto-orphism $\psi$ is almost regular. By using Thompson's classic theorem, if $\psi$ is a regular auto-Orphism (or equally $C_x(\phi) = 1$), $\mathcal{R}$ is nilpotent and we see clearly that if $\psi$ is almost regular, $\mathcal{R}$ must be almost nilpotent. In the other words, if $n = |C_x(\phi)|$, $\mathcal{R}$ is a nilpotent subring of the bounded index in terms of $p$ and $n$. This result is a combination of Fong, Hartley, Meixner, and Pettet works. This paper concludes that auto-Orphism $\phi$ is "almost regular", if there is a restriction on the rank $C_x(\phi)$ and $\mathcal{R}$ is "almost nilpotent" when any auto-Orphism $\psi$ is almost regular.

Key words: Automorphism; Finite ring; Regular; Soluble; Nilpotent; Hall–Higman-type theorems; Powerful

1. Introduction

By using Higman theorem, we know that nilpotency class of nilpotent rings with automorphisms of first prime order $p$ is $p$-bounded (Higman, 1957). By using Khukhro theorem, if ring $\mathcal{R}$ is nilpotent and $n = |C_x(\phi)|$, $\mathcal{R}$ has a subring of the $(p,n)$-bounded index whose nilpotency class is $p$-bounded (Khukhro, 2000). So if $\mathcal{R}$ is nilpotent, $C_x(\phi)$ is of $p$-bounded rank. Then $\mathcal{R}$ must have a subring of $(p,n)$-bounded coprime whose nilpotency class is $p$-bounded and has a subring of the $(p,n)$-bounded nilpotency class with a quotient of $(p,n)$-bounded rank.

Theorem (a): Assume that $\mathcal{R}$ is a $p$-ring that has an auto-Orphism $\phi$ of the first prime order $p$ such that $C_x(\phi)$ has the rank $n$. Consider that $S(\mathcal{R})$ is the soluble radical of $\mathcal{R}$, and then $\mathcal{R}$ is a $(p,n)$-bounded quotient of $S(\mathcal{R})$. So, $\mathcal{R}$ has $R' \leq N \leq R$ characteristic subrings such that $N / R'$ is nilpotent and $R / N$ and $R$ have $(p,n)$-bounded rank.

Theorem (b): Assume that the finite ring $\mathcal{R}$ has an auto-Orphism $\phi$ of the first prime order $p$ with the subring $C_x(\phi)$ of the rank $n$. So, $\mathcal{R}$ has $R' \leq N \leq R$ characteristic subrings such that $N / R'$ is nilpotent and $R / N$ and $R$ have bounded $(p,n)$-rank.

Result: Consider that $\mathcal{R}$ is a locally soluble finite ring and $x$ is an element of $p$ order when $C_x(\phi)$ is of rank $n$. Then $\mathcal{R}$ has $R' \leq N \leq R$ subrings that $N / R'$ is locally nilpotent and $R / N$ and $R$ have bounded $(p,n)$-rank.

Theorem 1: Consider $\mathcal{M}$ as a subring of the finite ring $\mathcal{R}$, if $\mathcal{M}$ is the direct product of simple non-commutative subrings $R_1, R_2, \ldots, R_m$ is a direct product of $\mathcal{R}$ minimal subrings and particularly $n = \chi(\mathcal{R})$.

Proposition 2-3: Consider that $\mathcal{R}$ is a simple ring with an auto-Orphism $\phi$ of the first prime order $p$ such that $C_x(\phi)$ is of rank $n$, $\mathcal{R}$ is a soluble radical of $\mathcal{R}$ is $(p,n)$-bounded.

Lemma 3-3: Assume that $\mathcal{R}$ is a $p$-ring with an auto-Orphism $\phi$ of the first prime order $p$ such that $C_x(\phi)$ is of rank $n$, then $\mathcal{R}$ is a $(p,n)$-bounded.

Proposition 2-3: Consider that $\mathcal{R}$ is a $p$-ring with an auto-Orphism $\phi$ of the first prime order $p$ such that $C_x(\phi)$ is of rank $n$, then $\mathcal{R}$ is a $(p,n)$-bounded.

2. Hall–Higman result

In this section, we mention Hall–Higman theorem that is highly applied in studying finite rings theory.

Lemma 3-5: Assume that $\tau \leftrightarrow \phi$ is a semi-direct product of $\mathcal{R}$, subring of $\mathcal{R}$, and a $\phi \leftrightarrow \phi$ of order $p$, and $p$ and $\tau$ are separate first primes. Consider that $\alpha (\mathcal{R}) = t$ and $\frac{\tau}{\alpha}$ are commutative and of potente $\mathcal{R}$. Consider that $\mathcal{R} \leftrightarrow \phi$ acts faithfully with linear conversions on vector space $V$ on a $\mathcal{R}$ or $t$ with characteristic $K$ of a close algebraically field. If $\dim(V), C_x(\phi)$ is of rank $t$, rank $T$ has become bounded in terms of $p$, $\tau$, and $\pi$.

Theorem 3-6: Consider that $\alpha \leftrightarrow \alpha$ is a semi direct product of $\mathcal{R}$ of a class equal or smaller than $2$ with a ring $\alpha \leftrightarrow \alpha$ of the first prime order $\mathcal{R}$ of $\mathcal{R}$. Now assume that:

\[
\begin{align*}
(1) \quad & \frac{H}{\alpha} \text{ is of potente } p \\
(2) \quad & H = [H, \alpha] \neq 1
\end{align*}
\]
Now consider $K$ as a close algebraically field of characteristic $k$. Consider $r, s, v$ and $v$ as a faithful $k$-modulus. Then

(a) $H$ is extraordinarily special and $Z(H) \leq Z(R)$ and $H \not\leq 1$ has become bounded by using function $\tau(d, q)$ that $d = d_{\max} C_f(A)$.

(b) Proposition 1-4: Assume that $R$ is a finite solvable $\phi$-ring that has an auto-adjoint $\phi$ of the first prime order $\phi$ and $F(\phi)$ is of rank $F$. If $R = R(\phi)$ and $L$ is a $\phi$-admissible $\phi$-subring of $R$ that $F(\phi)$ is of odd order, then $\tau(F(L))$ subring has bounded $(p, n)$-rank.

Proof: Assume that $\phi$ is a first prime that counts order $F(L)$ that $L$ is a reliable $\phi$-subring of ring $R$ and also consider that $\pi = \{2, q\}$ is a set of first primes. Consider that $\phi$ is a Hall $\phi$-subring of $\pi(N)$ and $\pi$ is a Hall $\pi$-subring of ring $R$. We consider $\phi = \{2, q\} = \{1\}$, where $\pi(N) = \{1\}$ is in ring $R$, $\pi$ is a Hall $\pi$-subring of ring $R$. Because $\tau(\pi(N))$ is in ring $R$, we can assume that $\pi(N)$ is a subring of $\pi$. Thus $\pi(N)$ is a subring of $N$ and $\pi$ is is a Hall $\pi$-subring of $N$. Then $K \cap N \subseteq H^X \cap N^X = (H \cap N)^X = H_1 \cap N \supseteq K \cap N^X$.

(1) $K \cap N \subseteq H^X \cap N^X \Rightarrow H_1 \cap N \supseteq K \cap N^X$.

(2) $|H^X \cap N^X| = |H_1 \cap N| \supseteq |K \cap N^X|$.

So we conclude that $K = H \cap N$, $H_1 = K \cap N$, $H = K \cap N$ is a Hall $\pi$-subring of $N$. On the other hand, $FK(L)$ is of an odd order, so $\phi$ is a Hall $\phi$-subring of $FK(L)$. Hence, $\tau(\phi(L))$ on $\phi(L)$ is according to the act of Hall $\phi$-subring of $FK(L)$ because $\phi(F(L))$ is in ring $R$. We consider $\phi \in \{2, q\} = \{1\}$, where $\phi = \{1\}$ is in ring $R$, $\phi$ is a Hall $\phi$-subring of ring $R$. Because $\phi \in \{2, q\}$ is in ring $R$, we have $\phi = \{1\}$.

So, $[F(L), \phi] = [P, Q] = 1 \Rightarrow [P, Q] = 1$. Since $P = \{1, 2\}$, we have $\phi(\pi(N)) = 1$. So, $\pi(N)$ is a Hall $\pi$-subring of $N$. Then $\pi(N)$ is a subring of $\pi$. Because $\pi(\pi(N))$ is in ring $R$, we can assume that $\pi(\pi(N))$ is a subring of $\pi$. Thus $\pi(\pi(N))$ is a subring of $\pi$. We consider that by using lemma 2-18, $\phi(N)$ is also a reliable $\phi$-subring of ring $R$.

Proof of Proposition 1-4:

Assume that $\phi(N) = \pi(\pi(N))$, then $[M, M] = [\pi(\pi(N)), \pi(\pi(N))] = (2f + 1) - (2f + 1) = 0 = 1$.  So, $M = \pi(\pi(N))$ is an nilpotent and of class 4 $\pi(\pi(N))$; because $\phi(N)$ is produced by a number of $(p, r)$-bounded numbers. So, $M$ is nilpotent and of $(p, r)$-bounded class and $\tau(\pi(N))$ rank is also $(p, r)$-bounded. Particularly $\tau(\pi(N))$ rank is also $(p, r)$-bounded. So, a powerful subring of $M = \tau(\pi(N))$, is also $(p, r)$-bounded. So, $\tau(\pi(N))$ rank is $(p, r)$-bounded and so it is finished.

Proposition 1-5: Assume that $R$ is a finite solvable $\phi$-ring that has an auto-adjoint $\phi$ of the first prime order $\pi$ such that $c_\phi(\phi)$ is of rank $F$. Then there is a series $R > N > R'$ that has $(p, r)$-bounded ranks.

Now assume that $H = O_{\pi + 1}(R \phi)$ so that we define semi direct product of $2$-nilpotent subring and $\phi$-rings $\phi$ and $\phi$ as a $\phi$-reliable subring of $H$.

We consider $\beta$ as the set of all $G(\phi)$-reliable parts $V$ of $O_{\pi + 1}(H)$ that are $\pi$-primary commutative subrings that have 2 conditions:

(1) $V$ is a combination set of $G(\phi)$.

(2) $C_V(\phi) = 1$.

Now we define $K$ as below:

$K = \bigcap_{v \in B} C_H(V)$.

That is a reliable $v$-subring of ring $R$. We know that $O_{\pi + 1}(H)$ is nilpotent. So by using condition (1), any $V \in B$ in $O_{\pi + 1}(H)$ is central. So, $C_H(V) \geq O_{\pi + 1}(H)$ and $C_H(V) \geq O_{\pi + 1}(H) C_H(V)$. Because $H = O_{\pi + 1}(H)$ $W$ that $W$, as introduced earlier, is a $\phi$-reliable subring of $2$-nilpotent subring of $H$. So $K = O_{\pi + 1}(H) \cap \bigcap_{v \in B} C_H(V)$.

3. Theorem 2-5:

1- If $\pi$ is maximum rank of subrings of finite solvable ring $R$, rank of this ring is $d + 1$ at most.

2- If $d$ is maximum rank of subrings of finite ring $R$, rank of this ring is $2d$ at most.

Lemma 3-5: Rank of $Y(\pi, k)$ is $(p, r)$-bounded.

Argument: By using theorem 2-5 of the first part (12), it is enough, for first prime $\pi \\ 2$, to calculate $\pi - (\pi, k)$ rank that $\phi$ is a $\phi$-subring of $FK(L)$ and $Y = W \cap K$ is a $\phi$-reliable subring of $K$.

(1) $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpotent. We consider that $\tau(\phi(\phi))$ is 2-nilpoten
That all $M \cdot s$ are reliable $\varphi$-subrings of $R$. Now assume that $\varphi$-primary factors of this series and since this series is central in $\phi$, all $V \cdot s$ are $\varphi$-primary commutative rings and can be considered as $F_{(N)} \subseteq \gcd$-module. Consider for a number of $i$ that $[V_{(N)}, V] \neq 0$. Since $[V_{(N)}, Y] = Y_{(N)}(V_{(N)}, K)$, $V_{(N)}$, and $K$ are $\varphi$-reliable, $[V_{(N)}, Y]$ is $\varphi$-reliable too. If $U \subseteq U$ is a $\varphi$-reduced sub-module of $[V, Y]$, we have $C_{(N)}(V) \neq 0$ with $K \geq Y$. So, $C_{(N)}(V) \neq 0$ when $[V_{(N)}, Y] = 0$. Therefore, according to (23) for a number of $k \leq 2 f + 1$, we have $[[V_{(N)}, Y] = 0$ and $[V_{(N)}, Y] = 0$. In other words, we have:

$[[Y, V, M, 1, Q, 1] \leq [[M, 1, Q, 1] \leq M_{11}$

$M_{11} \supseteq (M_{1}, Q, 1) \subseteq M_{11}$. So, $M_{11} \supseteq (M_{1}, Q, 1)$ is a powerful $\varphi$-subring of $Q$. The quotient ring $\frac{M_{11} \sum_{(M_{1}, Q, 1)}}{M_{11}}$ is nilpotent of class $i + 1$ because $Q_{i}$ is produced by $(p, r)$-bounded and $\frac{M_{11} \sum_{(M_{1}, Q, 1)}}{M_{11}}$ is nilpotent of $(p, r)$-bounded class. $\frac{M_{11} \sum_{(M_{1}, Q, 1)}}{M_{11}}$ has been $(p, r)$-bounded too which is according to $i$-rank of powerful subring $\frac{M_{11} \sum_{(M_{1}, Q, 1)}}{M_{11}}$. So, $Q_{i}$ is $(p, r)$-bounded and this is exactly what we need.

Theorem 1-6: If finite soluble ring $R$ has a subrings $R' \subseteq N \subseteq R$ that $\frac{N}{R}$ is nilpotent, $\frac{N}{R}$ and $\frac{N}{R}$ are of rank $F$, $R$ has subrings $R_{i} \subseteq N, R_{i} \subseteq R$ that $\frac{N}{R_{i}}$ is nilpotent, $\frac{N}{R_{i}}$ and $\frac{N}{R_{i}}$ have $r$-bounded rank.

Proof: Any soluble ring of rank $R$ has $r$-bounded fitting height, height of $N$ is $R$-bounded. We construct proof on this height by induction. If $N$ is nilpotent, we can assume that $N_{i} = F(R)$ and $R_{i} = 1$ and it is finished.

Now we claim that $N$ is not nilpotent. We study $S = N_{i} = F(R)$ that is a nilpotent subring of $R$ included in $F(R)$. For briefness we set $A = Aut(R)$. So it suffices to prove that $S_{A} = \prod_{a \in A} S_{A}$ has $r$-bounded rank. So induction assumption with $\frac{N}{R}$ as a characteristic subring can be applied. Now by using theorem 2-5 of part 1, it suffices to obtain $q$-rank of subrings of $S_{A}$ for any $q$-first prime. Assume that $\varphi$ is a $s$-subrings of $S$. Then $q_{\varphi} = \prod_{\varphi \in \varphi} S_{\varphi}$ is a $\varphi$-subring of $S_{A}$. Assume that $\varphi$ is $p$-subring of $S_{A}$, Then $q_{\varphi} = \prod_{\varphi \in \varphi} S_{\varphi}$ is a $\varphi$-subring of $S_{A}$. So it suffices to prove that rank $V$ is $r$-bounded. So we can select a $rK$-bound from automorphisms $a_{1}, a_{2}, \ldots, a_{n} \in A$ such that

$$\prod_{a \in A} \varphi(a) \cdot \varphi(a) = V = \varphi(a) \cdot \varphi(a)$$

So by using basic Burnside theorem, we have

$$\prod_{a \in A} Q_{a} = Q_{a} \quad \text{rank of this ring equals } r_{a} \quad \text{at most because } Q_{a} \leq R.$$ By using attention 5 of part $q$, we have $Q_{a} = \prod_{a_{1}, a_{2}, \ldots, a_{n} \in A} Q_{a}$, that $S_{r}$ is either a $l$-subring of $F_{(N)}$ or a $l$-subring of $\frac{N}{R}$. Now since $\frac{r}{R}$ equals $r$ at most, we can select $r$-bounded prime from first primes $t_{1}, t_{2}, \ldots, t_{n}$ that are separate from $q$ and elements $l_{i}$ and $s_{i}$ are subrings of $\frac{N}{R}$ such that

$$\frac{Q}{\varphi} = \prod_{i \in S} \left( \frac{Q}{\varphi} \cdot h_{i} \right).$$

Then

$$V = \prod_{i \in S} \left[ \prod_{j \in S} \left( \frac{Q}{\varphi} \cdot h_{i} \right) \right]$$

We note that for each $i$, the image of subring $\frac{V_{i}}{N}$ in $\frac{N}{R}$ is included in $i$-subring $T$. of $\frac{N}{R}$. We claim that rank $T$ is $r$-bounded. On $\frac{N}{R}$, rank of $T \cap N$ is less than or equal to $r$. Now consider $N$ as image of $N$ in $\frac{N}{R}$. So similarly, $\frac{N}{R}$ rank is less than or equal to $r$. Since $i \neq q$ and $\frac{N}{R}$ is nilpotent, so the subring $[T \cap N, V \cap N]$ is included in $i$-subring $T$. of $\frac{N}{R}$. So rank is less than or equal to $r$. Now by using Masch theorem, $\frac{r}{r} \cap N$ rank is equal to $r$. As a result, $\frac{N}{R}$ rank is less than or equal to $2r$. So, $[T \cap N]$ rank is less than or equal to $2r$. Now since $T \cap N$ acts on $[V, T \cap N]$ evidently, $T \cap N$ rank is $r$-bounded by using theorem 2-13. So, $T$ rank is $r$-bounded too. Therefore, $[V, h_{i}]$ rank is less than or equal to $2r$ and for each $a_{1}, a_{2}, \ldots, a_{n} \in A$, $h_{i}$ is $r$-subring of $S_{A}$ that has $r$-bounded rank. Now by using basic Burnside theorem, we can select a $r$-bounded $r$ number $n_{1}$ among $h_{i}$ numbers that generates $r$-subring of $h_{i}, k = 1, 2, 3, \ldots, n_{1}$. Now since rank of each $[V, h_{i}]$ equals $2r$, rank of each $[V, h_{i}] = \prod_{i \in S} \left( [V, h_{i}] \right)$ exactly equals $2r$. Now by summing up on the $r$-bounded first
primes of \( l \) in (1), \( v \) rank becomes \( R \)-bounded and this is exactly what we needed.

Theorem (a): Assume that \( R \) is a \( P \)-ring that has an auto Orphism \( \Phi \) of the first prime order \( P \) such that \( C_{s}(P) \) has rank \( r \). consider that \( S(R) \) is soluble radical of \( R \), so \( S(R) \) quotient has \((p,n)\)-bounded rank. Hence \( R \) has \( \Phi \) that is locally nilpotent and \( N \)-ring. Then \( R \) and \( \Phi \) have \((p,n)\)-bounded rank.

Proof: By using proposition (1-5) for \( S(R) \), we see that there is a correspondence between subrings of \( S(R) \). Now theorem (c) lets us replace \( S(R) \) subrings with \( S(R) \) and \( R \) characteristic subrings and this completes the proof.

Theorem b: By using theorem (a) with \( O_{S}(R) \), we obtain characteristic subrings with the required features that these subrings are characteristic in \( R \). So it is enough to show that \( S(R) \) has \((p,n)\)-bounded rank. We know that rank of each \( P \) subring of \( R \) is \((p,n)\)-bounded. Consider the ring \( R = \sum p \cdot \frac{O_{S}(R)}{R} \). Ring \( O_{S}(R) \) acts faithfully with its conjugates on \( \Phi \) that is a commutative ring with \( p \) potent and \((p,n)\)-bounded rank. So, by using basic Burnside theorem, \( \frac{O_{S}(R)}{R} \) rank is \((p,n)\)-bounded rank too. So, \( \frac{O_{S}(R)}{R} \) rank is also \((p,n)\)-bounded. As a result, \( \frac{O_{S}(R)}{R} \) rank is \((p,n)\)-bounded. This completes the proof.

4. Result

Assume that \( R \) is a locally finite solvable ring and \( g \) is an element of \( P \) order that \( C_{s}(R) \) is of rank \( R \). Then \( R \) has \( N \leq N \leq R \) subrings that \( N \) is locally nilpotent and \( N \) have \((p,n)\)-bounded rank.

Proof: We consider \( \Sigma \) as the family of all finite subrings of ring \( R \) that includes \( \varepsilon \). For each \( H \in \Sigma \), we consider \( S_{n} \) as the set of all pairs \( (N,R) \) of \( N \leq N \leq R \) subrings such that \( \frac{N}{R} \) is nilpotent and \( N \) and \( R \) have ranks smaller than or equal to \( (p,n) \) which is the function obtained from theorem 2-6. Now we apply theorem (b) by \( H \) and internal auto Orphism by \( \varepsilon \). It is clear that each \( S_{n} \) is finite. For both \( H_{1}, H_{2} \in \Sigma \) that \( H_{1} \geq H_{2} \), \( S_{1} \geq S_{2} \) exists. This converse system of finite sets has finite limited converse. Existence of subrings corresponding with \( N \) and \( R \) on each element of limited converse gives the required subring of ring \( R \).

References
