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Exact Jackson-Stechkin inequality in $L_2(R^2, e^{-x^2-y^2})$

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Abstract: Jackson-Stechkin exact inequality in $L_2\left(R^2, e^{-x^2-y^2}\right)$. M. G. Esmaganbet. In the given work for functions integrated with a square in all plane with weight $e^{-x^2-y^2}$ Jackson- Stechkin new exact inequalities are received when values of the module of smoothness of differentiated functions undertake in any point. From results of work follows, that in questions of definition of exact constants for functions integrated with a square in all plane with weight $e^{-x^2-y^2}$ algebraic polynoms, which spectra lay in a circle, a triangle and a hyperbole play the same role, that algebraic polynoms in an one-dimensional case.

Key words: *2π-periodic function; A trigonometric polynomial constants in inequalities of Jackson-Stechkin; Modulus of smoothness of functions; Inequality; The function of many variables; Arbitrary point spherical point; Triangular point; Hyperbolic point; The algebraic polynomial subspace*

1. Introduction

When 2π-periodic functions by trigonometric polynomials approximate the definition of exact constants in Jackson-Stechkin inequality in the fixed values of the modulus of smoothness of functions involved in many izvestnyematematiki: N.P.Korneychuk, N.I.Chernyh, VIBerdyshev, V.V.Zhuk , L.V.Taykov, V.V.Arestov, A.A.Ligun, VA Yudin, V.I.Ivanov and others.

To date, the subject most thoroughly studied approximation of square integrable 2π-periodic functions by trigonometric polynomials. Although in this case, the question remains about the accuracy of the Jackson-Stechkin inequality for differentiable functions when the values of the modulus of smoothness of functions are not fixed, arbitrary, as well as determine the exact constants in the approximation of functions of several variables.

In this paper, the functions square-integrable on the whole plane with weight $e^{-x^2-y^2}$ obtained new exact Jackson-Stechkin inequality when the values of the modulus of smoothness of differentiable functions are taken at an arbitrary point.

1. $L_2\left(R^2, e^{-x^2-y^2},\right) \equiv L_2*$ Let the space of functions with finite norm $f(x,y)$

$$\|f\|_* = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x,y)|^2 e^{-x^2-y^2} dxdy \right)^{\frac{1}{2}} < \infty$$

Across:

$$\overset{\Delta}{E}_n(f) = \inf_{\overset{\Delta}{p}_n \in \overset{\Delta}{M}_n} \left\| f - \overset{\Delta}{p}_n \right\| *,$$

$$\overset{\circ}{E}_n(f) = \inf_{\overset{\circ}{p}_n \in \overset{\circ}{M}_n} \left\| f - \overset{\circ}{p}_n \right\| *,$$

$$\tilde{E}_n(f) = \inf_{\tilde{p}_n \in \tilde{M}_n} \left\| f - \tilde{p}_n \right\| *$$

Denote triangular, spherical and hyperbolic best approximation of by algebraic polynomials of the form:

$$\overset{\Delta}{p}_n(x,y) = \sum_{0 \le k+l \le n} a_{kl} \cdot x^k \cdot y^l \tag{1.1}$$

$$\overset{\circ}{p}_n(x,y) = \sum_{\sqrt{k^2+l^2} \le n} a_{kl} \cdot x^k \cdot y^l \tag{1.2}$$

$$\tilde{p}_n(x,y) = \sum_{k \cdot l \le n} a_{kl} \cdot x^k \cdot y^l \tag{1.3}$$

Where $\overset{\Delta}{M}_n$, $\overset{\circ}{M}_n$, $\tilde{M}_n$ - subspace of algebraic polynomials of the form (1.1) - (1.3) respectively. A $Y_{n,m}(f)$, $E_{n,m}(f)$ denote the best

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approximation of "angle" and complete the best approximation by algebraic polynomials.

The shift is defined as follows:

$$\left(\tau_h^1\left(\tau_\eta^1 f\right)\right)(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(x\sqrt{1-h^2}+hu,\ y\sqrt{1-\eta^2}+\eta v\right)e^{-u^2-v^2}dudv$$

$$\left(\tau_h^m\left(\tau_\eta^n f\right)\right)(x,y)=\left(\tau_h^1\left(\tau_h^{m-1}\left(\tau_\eta^1\left(\tau_\eta^{n-1} f\right)\right)\right)\right)(x,y)$$

$m,\ n=1,2,\ldots$

Overall, a mixed modulus of smoothness and complete in the direction of the bisectors of the 1st and 3rd quadrants define respectively.

$$\omega_r(f;\delta,\sigma)=\sup_{\substack{0\le h\le\delta\\0\le\eta\le\sigma}}\left\|\Delta_{h\eta}^r f\right\|^*=\sup_{\substack{0\le h\le\delta\\0\le\eta\le\sigma}}\left\|\sum_{k=0}^{\infty}(-1)^{[r]+k}\binom{r}{k}\tau_h^k\left(\tau_\eta^k f\right)(x,y)\right\|^* . \quad (1.4)$$

$$\omega_{r,\theta}(f;\delta,\sigma)=\sup_{\substack{0\le h\le\delta\\0\le\eta\le\sigma}}\left\|\Delta_h^r\,\Delta_\eta^\theta f\right\|^*=$$

$$=\sup_{\substack{0\le h\le\delta\\0\le\eta\le\sigma}}\left\|\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{[r]+[\theta]+k+l}\binom{r}{k}\binom{\theta}{l}\tau_h^k\left(\tau_\eta^l f\right)(x,y)\right\|^* . \quad (1.5)$$

$$\overset{\Delta}{\omega}_r(f;\delta)=\sup_{0\le h\le\delta}\left\|\Delta_{hh}^r f\right\|^*=\sup_{0\le h\le\delta}\left\|\sum_{k=0}^{\infty}(-1)^{[r]+k}\binom{r}{k}\tau_h^k\left(\tau_h^k f\right)(x,y)\right\|^* . \quad (1.6)$$

We introduce the operators:

$$D_x=\frac{\partial^2}{\partial x^2}-2x\frac{\partial}{\partial x},\ D_y=\frac{\partial^2}{\partial y^2}-2y\frac{\partial}{\partial y},$$

$$D=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-2x\frac{\partial}{\partial x}-2y\frac{\partial}{\partial y}=D_x+D_y$$

$$,\ \tilde{D}=D_y(D_x),\ \overline{D}=D_x(D_x)+D_y(D_y).$$

$L_2^\alpha*\left(\tilde{L}_2^\alpha*,\ \overline{L}_2^\alpha*\right)$ $(\alpha=1,2,\ldots)$ – class $f(x,y)$ functions having weak derivatives $\frac{\partial^k f}{\partial x^i\partial y^j}, i+j=k,\ k=0,1,2,\ldots 2\alpha$ $(k=0,1,2,\ldots 4\alpha)$ belonging to the space: $L_2*;D^0 f\equiv f,$ $D^\alpha f=D(D^{\alpha-1}f)\in L_2*,$ $(\tilde{D}^\alpha f,\ \overline{D}^\alpha f\in L_2*)$.

Let:

$$H_n(x)=n!\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k}{k!}\cdot\frac{(2x)^{n-2k}}{(n-2k)!},\ (n=0,1,2,3,\ldots)$$

Hermite polynomials ortho normal system [12, p. 114] and

$$C_{nm}(f)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-x^2-y^2}H_n(x)\cdot H_m(y)dxdy$$

Fourier coefficients Ermita on the system $H_n(x)$ and

$$f(x,y)\sim\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}C_{kl}(f)\cdot H_k(x)\cdot H_l(y) \quad (1.7)$$

– Double row of Fourier-Hermite $f\in L_2*$,

$$\overset{\Delta}{S}_n(f;x,y)=\sum_{0\le k+l\le n}C_{kl}(f)\cdot H_k(x)\cdot H_l(y),$$

$$S_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{l=0}^{m}C_{kl}(f)\cdot H_k(x)\cdot H_l(y),$$

$$\overset{\circ}{S}_n(f;x,y)=\sum_{\sqrt{k^2+l^2}\le n}C_{kl}(f)\cdot H_k(x)\cdot H_l(y),$$

$$\tilde{S}_n(f;x,y)=\sum_{k\cdot l\le n}C_{kl}(f)\cdot H_k(x)\cdot H_l(y)$$

Triangular, rectangular, spherical and hyperbolic partial sums of Fourier-Hermite (1.7); the following three lemmas are proved in (Zhanys et al., 2012).

1.1. Lemma 1.1

Let L_{2^*} and has a number of (1.7), in the sense of convergence space 2 is equality:

$$\tau_h^1(\tau_\eta^1 f)(x,y) \overset{L_2^*}{=} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} C_{k,l}(f)(1-h^2)^{\frac{k}{2}}(1-\eta^2)^{\frac{l}{2}} H_k(x)H_l(y)$$

1.2. Lemma 1.2

Let *r*, $\theta \geq 0$. If the function f (x, y) $\in L_2$ * has a double Fourier-Hermite series (1.7), in the sense of convergence in the space of L_2 * there is equality:

$$\Delta^r_{h\eta} f(x,y) \overset{L_2^*}{=} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{[r]}[1-(1-h^2)^{\frac{k}{2}}(1-\eta^2)^{\frac{l}{2}}]C_{kl}(f)H_k(x)H_l(y), \tag{1.8}$$

$$\Delta^r_n \Delta^\theta_\eta f(x,y) \overset{L_2^*}{=} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{[r]+[\theta]}[1-(1-h^2)^{\frac{k}{2}}]^r[1-(1-\eta^2)^{\frac{l}{2}}]^\theta C_{kl}(f)H_k(x)H_l(y). \tag{1.9}$$

1.3. Lemma 1.3

The function $\varphi(h,\theta) = 1-(1-h^2)^\theta$ increases in each variable $\theta \in (0,+\infty)$, $h \in [0,1]$.

2. Theorem

Let there $r \geq 0, n-1, \alpha = 0,1,2,\dots,\ 0 < \delta \leq 1$ and $D^\alpha f(x,y) \in L_2$ *. Then the following sharp inequalities:

$$\overset{\Delta}{E}_{n-1}(f) \leq \frac{\overset{\Delta}{\omega}_r(D^\alpha f;\delta)}{(2n)^\alpha\left[1-(1-\delta^2)^{\frac{n}{2}}\right]^r}, \tag{2.1}$$

$$\overset{\circ}{E}_{n-1}(f) \leq \frac{\overset{\Delta}{\omega}_r(D^\alpha f;\delta)}{(2n)^\alpha\left[1-(1-\delta^2)^{\frac{n}{2}}\right]^r}, \tag{2.2}$$

$$\tilde{E}_{n^2-1}(f) \leq \frac{\overset{\Delta}{\omega}_r(D^\alpha f;\delta)}{(4n)^\alpha\left[1-(1-\delta^2)^{\frac{n}{2}}\right]^r}. \tag{2.3}$$

Evidence; It is known that (Abilov. 1995):

$$C_{k,l}(D^\alpha f) = (-2)^\alpha (k+l)^\alpha C_{k,l}(f). \tag{2.4}$$

From this and (1.8) on the Parseval equality, we have:

$$\overset{\Delta}{E}{}^2_{n-1}(f) = \left\|f - \overset{\Delta}{S}_{n-1}(f)\right\|^2 * = \sum_{k+l\geq n}^{\infty} C^2_{k,l}(f), \tag{2.5}$$

$$\overset{o}{E}{}^2_{n-1}(f) = \left\|f - \overset{o}{S}_{n-1}(f)\right\|^2 * = \sum_{\sqrt{k^2+l^2}\geq n}^{\infty} C^2_{k,l}, \tag{2.6}$$

$$\tilde{E}{}^2_{n-1}(f) = \left\|f - \tilde{S}_{n-1}(f)\right\|^2 * = \sum_{k\,l\geq n}^{\infty} C^2_{k,l}(f), \tag{2.7}$$

$$\left\|\Delta^r_{hh} D^\alpha f\right\|^2 * = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[1-(1-h^2)^{\frac{k+l}{2}}\right]^{2r} C^2_{k,l}(D^\alpha f) =$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[1-(1-h^2)^{\frac{k+l}{2}}\right]^{2r}(2(k+l))^{2\alpha} C^2_{k,l}(f). \tag{2.8}$$

Then from Lemma 1.3 and (2.8) we can write the following chain of inequalities:

$$\left\|\Delta^r_{hh} D^\alpha f\right\|^2 * \geq \sum_{k+l\geq n}\left[1-(1-h^2)^{\frac{k+l}{2}}\right]^{2r}(2(k+l))^{2\alpha} C^2_{k,l}(f) \geq$$

$$\geq (2n)^{2\alpha}\left[1-(1-h^2)^{\frac{n}{2}}\right]^{2r}\sum_{k+l\geq n} C^2_{k,l}(f) = (2n)^{2\alpha}\left[1-(1-h^2)^{\frac{n}{2}}\right]^{2r}\overset{\Delta}{E}{}^2_{n-1}(f).$$

It remains to choose the supremum of the last inequality, by Lemma 1.3 and

$$\left\|\Delta^r_{\delta\delta} D^\alpha f\right\|^2 * \leq \sup_{0\leq h\leq\delta}\left\|\Delta^r_{hh} D^\alpha f\right\|^2 * = \overset{\Delta}{\omega}_r(D^\alpha f;\delta). \tag{2.9}$$

Applying inequality $k+l \geq \sqrt{k^2+l^2}$, $k+l \geq 2\sqrt{k\,l}$ with the help of (2.6) - (2.8) and Lemma 1.3, we have:

$$\left\|\Delta^r_{hh} D^\alpha f\right\|^2 * \geq \sum_{\sqrt{k^2+l^2}\geq n}\left[1-(1-h^2)^{\frac{\sqrt{k^2+l^2}}{2}}\right]^{2r}(2\sqrt{k^2+l^2})^{2\alpha}C^2_{k,l}(f)\geq$$

$$\geq (2n)^{2\alpha}\left[1-(1-h^2)^{\frac{n}{2}}\right]^{2r}\sum_{\sqrt{k^2+l^2}\geq n}C^2_{k,l}(f) = (2n)^{2\alpha}\left[1-(1-h^2)^{\frac{n}{2}}\right]^{2r}\overset{o}{E}{}^2_{n-1}(f) \quad ,$$

and

$$\left\|\Delta^r_{hh} D^\alpha f\right\|^2 * \geq \sum_{k\,l\geq n^2}\left[1-(1-h^2)^{\frac{\sqrt{k\,l}}{2}}\right]^{2r}(4\sqrt{k\,l})^{2\alpha}C^2_{k,l}(f)\geq$$

$$\geq (4n)^{2\alpha}\left[1-(1-h^2)^n\right]^{2r}\tilde{E}{}^2_{n^2-1}(f)\ .$$

As for the polynomial Ermita $H_n(x)$:

$$C_{k,l}(H_n) = \begin{cases}1, \ k=n, l=0\\ 0, \ k\neq n, \forall\, l\end{cases},$$

$$C_{k,l}(H_n(x)H_n(y)) = \begin{cases}1, \ k=n=l\\ 0, \ k\neq n, \forall\, l\end{cases}$$

(2.5) - (2.8) we have:

$$\overset{\Delta}{E}_{n-1}(H_n) = \overset{o}{E}_{n-1}(H_n) = 1 \quad , \qquad \left\|\Delta^r_{hh} D^\alpha H_n\right\|^* = (2n)^\alpha\left[1-(1-h^2)^{\frac{n}{2}}\right]^r \quad ,$$

$$\tilde{E}_{n^2-1}(H_n(x)H_n(y)) = 1 \quad , \qquad \left\|\Delta^r_{hh} D^\alpha H_n(x)H_n(y)\right\|^* = (4n)^\alpha\left[1-(1-h^2)^n\right]^r \ .$$

Therefore, by Lemma 1.3 for $f_0(x) = H_n(x)$ (2.1) - (2.3), and for $f_1(x, y) = H_n(x)H_n(y)$ (2.3) equalities.

3. Comment

VA Yudin From the results of (Yudin, 1981) that in matters of determination of exact constants in Jackson-Stechkin inequalities for 2p-periodic functions of several variables by trigonometric polynomials, spectra lie in a circle playing the same role that the trigonometric polynomials in the one-dimensional case.

For the function of a single variable is denoted by $E_{n-1}(f)$, $\omega_r(f,\delta)$- the best approximation by algebraic polynomials p (x) $= \sum_{k=0}^{n-1} a_k x^k$ and smoothness modulus of order r> 0 then there is a sharp inequalities

$$E^{n-1}(f) \leq \frac{\omega_r(D^\alpha{}_x f;\delta)}{(2n)^\alpha\left[1-(1-\delta^2)^{\frac{n}{2}}\right]^r}.$$

From this and from the theorem that in identifying the exact constants for functions square-integrable on the whole plane with the weight of algebraic polynomials, spectra lie in the circle; triangle and hyperbole are playing the same role as that of an algebraic polynomial in one dimension.

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